Response to Pressurization of a Viscoelastic Cylinder with an Eroding Internal Boundary

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A series solution is obtained for the stresses in a viscoelastic cylinder that has an eroding inner boundary acted upon by a pressure. The outer boundary is constrained by a thin elastic shell. The method of analysis utilizes an indirect application of the integral transform method normally employed in quasi-static viscoelastic analysis and allows an exact inversion even though the viscoelastic material characterization is specified in general form. The resulting triaxial stress conditions on the moving inner boundary for several particular cases are displayed.

Nomenclature

r	= radial measure in cylindrical coordinates
a_o	= initial inner radius
a	= moving inner radius
b	= outer radius
w	= radial displacement, cylindrical coordinates
p	= pressure on moving inner boundary
p_o	= equivalent pressure on fixed inner boundary
q	= interacting pressure between cylinder and case
h	= case thickness
E_c, ν_c	= Young's modulus and Poisson's ratio, respectively, of the case
G, ν	= shear modulus and Poisson's ratio, respectively
σ_r, σ_θ, σ_z	= normal stress components, cylindrical coordinates
t	= time
s	= Laplace transform parameter

Introduction

MOVING boundary problems in linear viscoelasticity have recently become of general interest, and this paper presents a new approach to such problems. The primary stimulation for this type of problem comes from the firing of solid-propellant rocket engines, wherein the coordinates of the inner boundary are continuously changing with time because of the burning of the propellant.

One aspect of the general response problem of solid rocket engines centers around the determination of the stress-strain fields resulting from the applied internal pressure and the relating of these quantities to failure criteria. In addition to the applied pressure, there is of course a thermal environment that not only induces thermal stresses but also couples with the pressurization response, since the idealized viscoelastic mechanical properties are functions of temperature. However, a simple heat-diffusion analysis with a realistic value of the thermal diffusivity reveals that the burning boundary moves much faster than the rate at which any significant amount of heat can be diffused; thus, all thermal effects are confined to the immediate region near the surface and can usually be neglected.

The determination of the mechanical response for moving boundary viscoelastic problems is generally much more difficult than for comparable fixed coordinates problems. Transform methods are immediately applicable to fixed boundary coordinates problems, wherein the Laplace (or Fourier) transformed solution is directly obtained from the transformed field equations, boundary conditions, and constitutive relations. However, the direct application of transform methods to moving boundary problems is not possible since the transform of the boundary conditions can be taken only at fixed (and not moving) coordinates points. A solution must then be sought directly from the governing viscoelastic field equations and boundary conditions. This procedure characteristically involves separate integrations with the space and time variables, parts of which integrations generally require numerical evaluation. This report presents and applies a general method of solving such moving boundary viscoelastic problems through the indirect application of the Laplace transformation. Before proceeding to describe and apply this method, it is instructive to review the previous moving boundary viscoelastic solutions and to outline briefly the development of the usage of transform methods in viscoelasticity.

The actual usage of transform methods in viscoelasticity was predated by the development of the analogy between elastic and viscoelastic problems. Alfrey[1] was apparently the first to note the analogy between the governing static elasticity and quasi-static viscoelasticity equations. He showed that the viscoelastic solution could be obtained directly from the elastic solution in cases where the viscoelastic problem could be solved by the separation of variables technique (separating the time and spatial parts of the solution). Later, Tsien[2] generalized Alfrey's work, still using the separation of variables. The first usage of integral transforms was given by Read.[3] Utilizing Fourier transforms, he showed that the viscoelastic problem could be reduced to one involving spatial dependence only, analogous to problems in elasticity. Later contributors, Sips,[4] Brull,[5] and Lee,[6] set up methods based upon the Laplace transform and various forms of the viscoelastic constitutive relations. In addition, Lee discussed certain conditions under which the use of transforms would not be applicable. Specifically, the direct application of transforms was restricted to problems in which the coordinates of the boundary do not change with time nor do the types of boundary condition (displacement or traction) at a boundary point change with time. Transform methods have been applied with success to obtain viscoelastic solutions of a wide class of problems. However, as discussed previously, transform methods cannot be applied directly to moving boundary problems. The moving boundary viscoelastic solutions that have necessarily been obtained by other means will now be reviewed.

Lee, Radok, and Woodward[7] obtained a solution for the pressurization of an annihilating viscoelastic cylinder con-

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strained by an elastic case. The analysis restricted the material to behave as a Kelvin model in shear and to be incompressible. The loading consisted of a step pressure increase. A similar analysis was performed by Arenz and Williams[8] for a sphere. Corneliussen and Lee[9] presented a solution for the pressurization of a spinning, annihilating, viscoelastic cylinder, where the outer boundary was stress free. Again the pressurization consisted of a step increase in pressure. The material behavior was assumed to be dilatationally elastic and to behave as a Maxwell model in shear. However, the portion of the solution given for σ_r, resulting from the pressurization, was found to be independent of the viscoelastic properties.

Corneliussen, Kamowitz, Lee, and Radok,[10] have presented a general solution for the pressurization of a cylinder including the effect of the moving boundary. However, the technique employed did not allow the inclusion of a constraining case on the outer boundary, and as a consequence the radial and tangential stresses were independent of the viscoelastic properties.

Shinozuka[11] has presented an analytic solution for a case bonded viscoelastic cylinder under pressurization. The material characterization was taken as that of the standard linear solid in shear, and incompressible. The pressure function consisted of an exponential form, and the time history of the coordinate of the moving (internal) boundary was given by a nonlinear relation. It does not appear to be practical to use Shinozuka's method to obtain solutions for problems with more complicated material characterizations.

Recognizing the difficulty of obtaining analytical solutions for more involved conditions than those assumed in the previous references, Rogers and Lee[12] have developed a suitable numerical integration scheme for such problems. Specifically they formulated the problem of the case bonded viscoelastic cylinder with general materials properties and arbitrarily varying internal pressure and inner boundary coordinate in terms of Volterra-type integral equations. These integral forms are then amenable to direct numerical integration.

Schapery[13] has developed a general method for solving moving boundary viscoelastic problems. In his approach, the moving boundary problem is replaced by a fictitious nonmoving boundary problem that is subjected to a time-dependent pressure. This pressure is to be determined such that the appropriate boundary-condition pressure exists on the moving boundary. This leads to an integral equation for determining the pressure on the fixed boundary problem that is solved by a method of successive approximations appropriate to convolution-type integral equations. The general method was illustrated on a simple example.

This report presents a method of solving moving boundary viscoelastic problems with comparable generality to the previously described method of Rogers and Lee, and of Schapery, but by a procedure that leads to an infinite series analytical solution for the problem, even for a general specification of the mechanical properties for the material. It may be noted that this is in contrast to previous methods of quasi-static viscoelastic analysis wherein for a specification of mechanical properties involving more than a few elements in a model representation, the procedure characteristically involves either an approximate transform inversion or a direct numerical integration of the governing equations.

Method of Solution

Consider the class of one-dimensional viscoelastic problems, with coordinate r, where the appropriate boundary conditions are that the stress be some specified function of time at a coordinate that is also a specified function of time and that a second condition be at some fixed coordinate for all values of time.

At $t = 0$ the coordinate of the moving boundary is given by $r = a_o$, and thereafter by $r = a(t)$. The appropriate boundary condition is then at $r = a(t)$

$$\sigma_r = p(t) \tag{1}$$

In a manner similar to that followed by Schapery,[13] hypothesize that, for an analogous problem with a fixed boundary coordinate at $r = a_o$, a pressure $p_o(t)$ can be applied at this coordinate such that (1) is satisfied. Consider $p_o(t)$ to be expressed by a complete set of functions $F_n(t)$

$$p_o(t) = \sum_{n=1}^{\infty} A_n F_n(t) \tag{2}$$

with as yet unknown multiplying constants A_n.

Using (2), obtain the exact solution to the fixed coordinate problem, by transform methods. Then, this solution can be written as

$$\sigma_r = \sum_{n=1}^{\infty} A_n f_n(r, t) \tag{3}$$

Evaluate (3) at $r = a(t)$ such that (1) is satisfied:

$$\sigma_r\big|_{r=a(t)} = p(t) = \sum_{n=1}^{\infty} A_n f_n (r, t)\big|_{r=a(t)} \tag{4}$$

Equation (4) is seen to involve the expansion of an arbitrary function $p(t)$ in terms of a set of nonorthogonal functions $f_n(r, t)|_{r=a(t)}$. One manner of adjusting the A_n coefficients such that (4) is satisfied would be through the minimization of the total square error; however, if $f_n(r, t)|_{r=a(t)}$ is an involved function of t, then the resulting integrations may be quite complicated. A somewhat simpler procedure is adopted here whereby $p(t)$ and $f_n(r, t)|_{r=a(t)}$ are each expanded in a complete set of orthogonal functions $Q_n(t)$. First, expanding the left side of (4),

$$p(t) = \sum_{m=1}^{\infty} B_m Q_m(t) \tag{5}$$

Expanding $f_n(r, t)|_{r=a(t)}$ in the series of $Q_m(t)$,

$$f_n(r, t)\big|_{r=a(t)} = \sum_{m=1}^{\infty} C_{mn} Q_m(t) \tag{6}$$

B_m and C_{mn} are determined utilizing the orthogonality of $Q_m(t)$. Substitution of (5) and (6) into (4) gives

$$\sum_{m=1}^{\infty} B_m Q_m(t) = \sum_{n=1}^{\infty} A_n \left[\sum_{m=1}^{\infty} C_{mn} Q_m(t) \right] \tag{7}$$

This will be satisfied by equating the coefficient of the general term

$$\sum_{n=1}^{\infty} C_{mn} A_n = B_m \tag{8}$$

Equation (8) can be written in matrix form where A_n is replaced by A_m, to be consistent with standard notation

$$[C_{mn}] \quad \{A_m\} = \{B_m\} \tag{9}$$

$$\{A_m\} = [C_{mn}]^{-1} \{B_m\} \tag{10}$$

This procedure gives an infinite number of equations to find A_m, but one would truncate this at some level with $m_{\max} = n_{\max}$. Proof that the resulting solution given by the right-hand side of (4) converges to $p(t)$ is dependent upon the proof that the functions $f_n(r, t)|_{r=a(t)}$ form a complete set. This completeness proof appears to be rather difficult to obtain, so as a matter of expediency it becomes necessary to demonstrate convergence in each particular problem under consideration. It should be emphasized however that this truncation procedure does not imply an approximate satisfaction of the governing differential equation of equilibrium, or the stress-strain relations, but rather only reflects approximations in the satisfaction of the boundary condition. Whereas convergence in problems involving the approximate satisfaction of the equations of equilibrium only can be demonstrated by taking an increasing number of terms and comparing the resulting solution with that obtained for fewer terms, in the present problem the degree of approximation can immediately be ascertained by comparing the given exact boundary condition $p(t)$ with the resulting approximate one $\sigma_r|_{r=a(t)}$. In fact, if at some particular number of terms $n_{\max}$ the difference between $\sigma_r|_{r=a(t)}$ and $p(t)$ is less than the degree of accuracy in the specification of the stress boundary condition $p(t)$, then $\sigma_r|_{r=a(t)}$ actually can be considered to be an appropriate boundary condition, and it follows that the resulting solution of the fixed boundary coordinate problem represents an exact closed form solution to the moving boundary coordinate problem. In practice it is convenient to take $\Sigma A_n F_n(t)$ and $\Sigma B_m Q_m(t)$ as Fourier series since the orthogonality and completeness of these are well known.

Analysis

Using bars over variables to designate Laplace transformed quantities the governing plane strain, axisymmetric equation of equilibrium in polar cylindrical coordinates is

$$\partial^2\bar{w}/\partial r^2 + (1/r)(\partial\bar{w}/\partial r) - \bar{w}/r^2 = 0 \tag{11}$$

The viscoelastic cylinder will be taken as being constrained by an elastic case. The thin elastic case, pressure-deflection relationship is

$$\bar{w}_c = \bar{q}\{[b^2(1-\nu_c^2)]/E_c h\} \tag{12}$$

The relevant stress-displacement relations are:

$$\bar{\sigma}_r = 2\bar{G}\left[\frac{\partial\bar{w}}{\partial_r} + \frac{\bar{\nu}}{1-2\bar{\nu}}\left(\frac{\partial\bar{w}}{\partial r} + \frac{\bar{w}}{r}\right)\right] \tag{13}$$

$$\bar{\sigma}_\theta = 2\bar{G}\left[\frac{\bar{w}}{r} + \frac{\bar{\nu}}{1-2\bar{\nu}}\left(\frac{\partial\bar{w}}{\partial r} + \frac{\bar{w}}{r}\right)\right] \tag{14}$$

$$\bar{\sigma}_z = 2\frac{\bar{G}\bar{\nu}}{1-2\bar{\nu}}\left(\frac{\partial\bar{w}}{\partial r} + \frac{\bar{w}}{r}\right) \tag{15}$$

In (13–15) $\bar{G}$ and $\bar{\nu}$ are the Laplace transformed operational shear modulus and Poisson's ratio, respectively, for the viscoelastic material of the cylinder.

Solving (11) subject to the boundary conditions that at $r = a_o$ a fixed coordinate $\bar{\sigma}_r = -\bar{p}_o$ and at $r = b$ the radial stress and displacement are continuous with those on the thin elastic case, it is found that

$$\frac{\bar{\sigma}_r}{\bar{p}_o} = \frac{-(\bar{K}-1) - (b^2/r^2)[(1-2\bar{\nu})\bar{K}+1]}{\bar{K} - 1 + (b^2/a_o^2)[(1-2\bar{\nu})\bar{K}+1]} \tag{16}$$

where

$$\bar{K} = E_c h/[2b(1-\bar{\nu}_c^2)\bar{G}] \tag{17}$$

Consistent with the procedure outlined in the previous section the stress at the fixed boundary coordinate is taken as

$$p_o(t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi t}{t_o}\right) \tag{18}$$

where A_n are as yet unknown coefficients. Transforming (18), $\bar{p}_o$ is given by

$$\bar{p}_o = \sum_{n=1}^{\infty} A_n \frac{(n\pi/t_o)}{s^2 + (n^2\pi^2/t_o^2)} \tag{19}$$

In what follows, Poisson's ratio will be taken as a real positive constant. This effectively implies that the viscoelastic bulk modulus differs from the shear modulus by a real positive constant factor. Although this is not strictly correct, it appears to be a reasonable assumption, for lack of more definitive information on the viscoelastic bulk modulus. However, the general method presented here can certainly be made to accommodate the general formulation on the bulk modulus, when the experimental data become available.

Taking the mathematical description of the viscoelastic shear modulus in the form of a completely general differential operator, the Laplace transformed shear stress-strain relation gives the transformed operator as

$$\bar{G} = [A(s)G_o]/B(s) \tag{20}$$

where $A(s)$ and $B(s)$ are nondimensional polynomials in the transform parameter s, the degree of which are determined by the number of terms retained in the differential operator form of the stress-strain relation, and G_o is the long time asymptotic relaxation modulus, factored out for later convenience.

Utilizing (16–20), the transformed stress $\bar{\sigma}_r$ can be written as

$$\bar{\sigma}_r = \sum_{n=1}^{\infty} \frac{[C(s) + (b^2/r^2)D(s)]A_n(n\pi/t_o)}{F(s)[s + i(n\pi/t_o)][s - i(n\pi/t_o)]} \tag{21}$$

where

$$C(s) = -E_c'B(s) + A(s)$$

$$D(s) = -(1-2\nu)E_c'B(s) - A(s)$$

$$F(s) = E_c'B(s) - A(s) + (b^2/a_o^2)[(1-2\nu)E_c'B(s) + A(s)]$$

$$E_c' = E_c h/[2b(1-\nu_c^2)G_o]$$

Since $F(s)$ is a polynomial in the transform parameter s, the roots can be determined numerically, and it can be written as

$$F(s) = F_0 \prod_{j=1}^{p} (s - a_j) \tag{22}$$

with p representing the degree of the polynomial $F(s)$.

Equation (21) can easily be inverted to yield

$$\sigma_r = \sum_{n=1}^{\infty} A_n\Bigg[\left(\gamma_{1n} + \gamma_{2n}\frac{b^2}{r^2}\right)\sin(n\pi t/t_o) + \left(\gamma_{3n} + \gamma_{4n}\frac{b^2}{r^2}\right)\cos\left(\frac{n\pi t}{t_o}\right) + \sum_{j=1}^{p}\left(\gamma_{5nj} + \gamma_{6nj}\frac{b^2}{r^2}\right)e^{a_j t}\Bigg] \tag{23}$$

where

$$\gamma_{1n} = \frac{C(-in\pi/t_o)}{2F(-in\pi/t_o)} + \frac{C(in\pi/t_o)}{2F(in\pi/t_o)}$$

$$\gamma_{2n} = \frac{D(-in\pi/t_o)}{2F(-in\pi/t_o)} + \frac{D(in\pi/t_o)}{2F(in\pi/t_o)}$$

$$\gamma_{3n} = \frac{iC(-in\pi/t_o)}{2F(-in\pi/t_o)} - \frac{iC(in\pi/t_o)}{2F(in\pi/t_o)}$$

$$\gamma_{4n} = \frac{iD(-in\pi/t_o)}{2F(-in\pi/t_o)} - \frac{iD(in\pi/t_o)}{2F(in\pi/t_o)}$$

$$\gamma_{5nj} = \frac{(n\pi/t_o)C(a_j)}{[F(s)/(s-a_j)]|_{s\to a_j}[a_j^2 + (n^2\pi^2/t_o^2)]}$$

$$\gamma_{6nj} = \frac{(n\pi/t_o)D(a_j)}{[F(s)/(s-a_j)]|_{s\to a_j}[a_j^2 + (n^2\pi^2/t_o^2)]}$$

Equation (23) represents the solution of the fixed boundary coordinate problem. The coefficients A_n are determined in the manner described in the preceding section to satisfy the boundary condition that at $r = a(t)$, $\sigma_r = p(t)$. Expanding $p(t)$ and (23) in a Fourier sine series and equating coefficients of the general term gives the terms in (9) as

$$B_m = 2\int_0^1 p(t_o T)\sin m\pi T\, dT \tag{24}$$

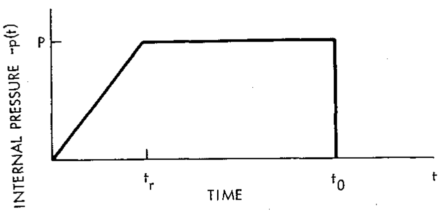

Fig. 1 Pressure acting on moving boundary.

and

$$C_{mn} = \gamma_{1n}\,\delta_{mn} + 2\,b^2 \int_0^1 \frac{(\gamma_{2n}\,\sin n\pi T + \gamma_{4n}\,\cos n\pi T)\,\sin m\pi T\,\mathrm{d}T}{a^2(t_o T)} + \sum_{j=1}^{p} \frac{\gamma_{5nj} 2m\pi\,(1 - \cos m\pi\, e^{a_j t_o})}{(a_j t_o)^2 + m^2\pi^2} + \sum_{j=1}^{p} 2\gamma_{6nj}\, b^2 \int_0^1 \frac{e^{a_j t_o T}\,\sin m\pi T\, dT}{a^2(t_o T)} + \frac{\Delta_{mn}\gamma_{3n}}{\pi}\left\{\frac{1}{(m-n)}\,[1 - \cos(m-n)\pi] + \frac{1}{(m+n)}\,[1 - \cos(m+n)\pi]\right\} \tag{25}$$

where

$$\begin{aligned} \delta_{mn} &= 1 \qquad m = n \\ &= 0 \qquad m \neq n \\ \Delta_{mn} &= 0 \qquad m = n \\ &= 1 \qquad m \neq n \end{aligned}$$

By specifying the time history of the moving boundary coordinate $a(t)$ and the pressure $p(t)$ on this boundary, B_m and C_{mn} can be evaluated, and the solution of (9) can be effected. Then the solution of the moving boundary problem is complete. Next a particular example will be considered, and the convergence character will be examined.

Application

The moving boundary coordinate will be taken as a linear function of time as

$$a(t) = a_o[1 + (t/t_1)] \tag{26}$$

The pressure acting on the moving boundary consists of a linear pressure rise followed by a steady pressure (Fig. 1).

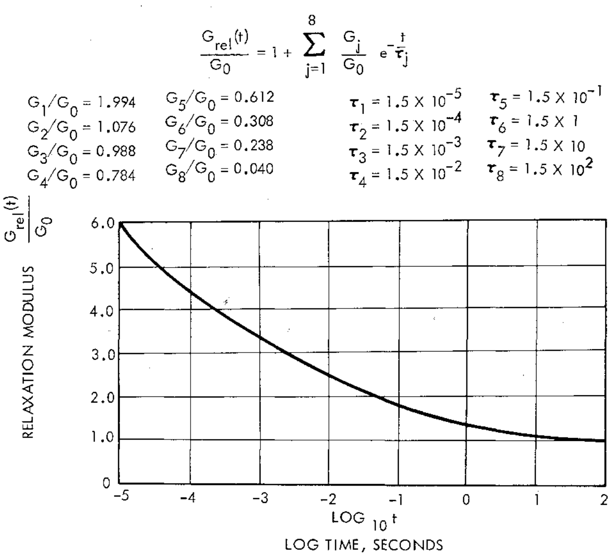

Fig. 2 Shear relaxation modulus.

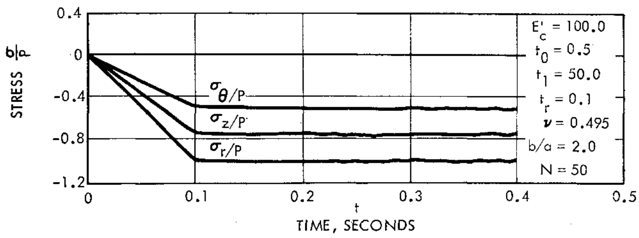

Fig. 3 Stresses on moving boundary, $r = a(t)$; $E_c' = 100$, $t_o = 0.5$.

The viscoelastic shear modulus will be taken as that determined by Gottenberg and Christensen[14] for a solid-propellant simulant composed of polyurethane matrix into which was embedded a finely divided dispersion of about 80% by weight of salt crystals and aluminum powder. The data given by this source are in the form of the shear relaxation modulus given over six decades of time, with the shortest time of 0.1 msec. It is of interest to note that this relaxation modulus was converted from the dynamic complex modulus that was determined by direct data reduction from test results up to a maximum frequency of 1000 cps. It has been shown by Fourier integral analysis that the input pulse shown in Fig. 1 (with $t_r = 0.01$ and $t_o = 0.05$) is very closely represented by a frequency content up to 1000 cps; thus the mechanical properties are specified in the appropriate range for this particular application. It is convenient to fit the shear relaxation modulus given in Ref. 14 by the exponential series

$$\underset{\text{rel}}{G(t)} = G_o + \sum_{j=1}^{8} G_j e^{-t/\tau_j} \tag{27}$$

The resulting values for G_j and τ_j along with the resulting relaxation modulus is shown in Fig. 2.

The relaxation modulus given by (27) is easily converted to the transformed differential operator modulus through the generalized Maxwell model formulation.[15] Thus the transformed shear stress-strain operator is given by

$$\bar{G} = G_o + \sum_{j=1}^{8} \frac{sG_j}{[s + 1/\tau_j]} \tag{28}$$

Through lengthy but straight forward algebraic manipulations, (28) can be reduced to the form given by (20), involving the ratio of polynomials in s.

Utilizing these data, the coefficients C_{mn} in (25) can be evaluated. Although the first integral in (25) can be expressed in terms of sine and cosine integrals, it is more convenient to leave it in the form shown and apply a direct numerical integration.

After selecting the number of terms to be retained in (18), say N, the system of algebraic equations (9) can be solved for the coefficients A_n. The radial stress σ_r follows from (23), and the other two stress components are readily shown to be

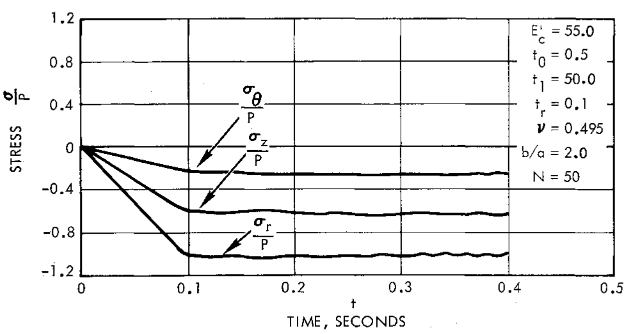

Fig. 4 Stresses on moving boundary, $r = a(t)$; $E_c' = 55$, $t_o = 0.5$.

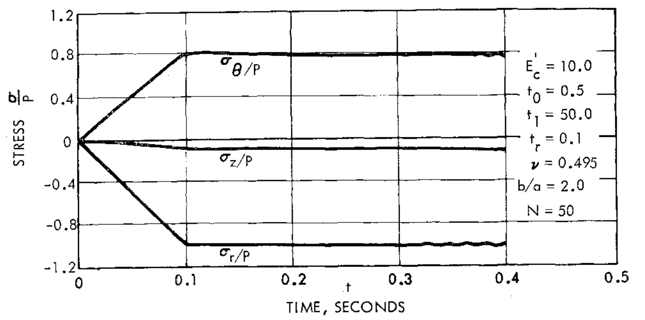

Fig. 5 Stresses on moving boundary, $r = a(t)$; $E_c' = 10$, $t_o = 0.5$.

given by,

$$\sigma_\theta = \sum_{n=1}^{N} A_n \left[\left(\gamma_{1n} - \gamma_{2n} \frac{b^2}{r^2} \right) \sin\frac{n\pi t}{t_o} + \left(\gamma_{3n} - \gamma_{4n} \frac{b^2}{r^2} \right) \cos\frac{n\pi t}{t_o} + \sum_{j=1}^{8} \left(\gamma_{5nj} - \gamma_{6nj} \frac{b^2}{r^2} \right) e^{a_j t} \right] \quad (29)$$

$$\sigma_z = 2\nu \sum_{n=1}^{8} A_n \left[\gamma_{1n} \sin\frac{n\pi t}{t_o} + \gamma_{3n} \cos\frac{n\pi t}{t_o} + \sum_{j=1}^{8} \gamma_{5nj} e^{a_j t} \right] \quad (30)$$

Particular numerical examples have been carried out, with the values of the relevant parameters taken corresponding to typical solid-rocket engines and with 50 terms retained in the series representation. Figures 3–5 show the triaxial stress conditions on the moving inner boundary for times in the immediate region of the initial rise time. The distinction between Figs. 3–5 is in changes in the relative stiffness of the case to the inner viscoelastic material. It can be seen from Figs. 3–5 that the resulting stress on the moving boundary $\sigma(r)|_{r=a(t)}$ is nearly the same as the given boundary condition, Fig. 1. Thus, the resulting analytical solutions for the boundary conditions shown in Figs. 3–5 are also effectively the solutions for the boundary condition of Fig. 1.

Figures 6 and 7 also reveal the effect of the relative stiffness of the case to the viscoelastic material but over the time range needed for the moving boundary to nearly reach the outer boundary. Contrary to the solutions shown in Figs. 3–5, the resulting boundary conditions in these cases are not exact replicas of that in Fig. 1. The reason for this approximation is that in the time scale taken here the rise time is almost an instantaneous occurrence, and the resulting near discontinuity places more stringent conditions of convergence upon the series solution. Nevertheless, these solutions would be expected to give reliable results except at very short times, in which time range the results in Figs. 3–5 are valid.

The effect of varying E_c', the effective stiffness ratio of the case to the viscoelastic material is well demonstrated in Figs. 3–7. For $E_c' = 100$, the resulting stresses on the moving internal boundary are approaching those of a hydrostatic

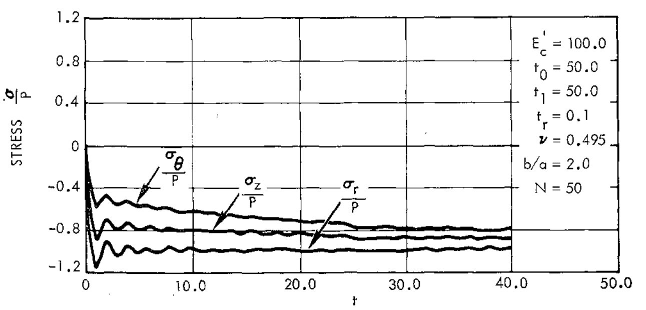

Fig. 6 Stresses on moving boundary, $r = a(t)$; $E_c' = 100$, $t_o = 50$.

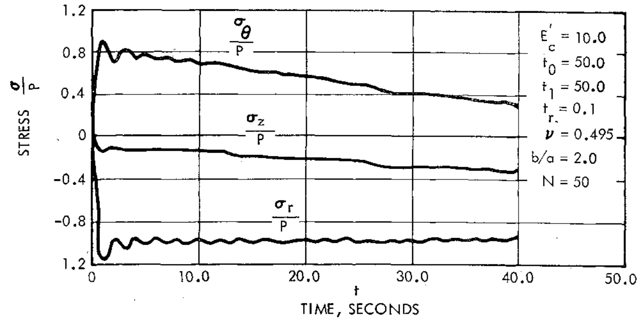

Fig. 7 Stresses on moving boundary, $r = a(t)$; $E_c' = 10$, $t_o = 50$.

condition whereas for $E_c' = 10$, the stresses on the moving boundary give rise to very large maximum shear stresses in addition to the large tensile hoop stress. In this latter case the stress conditions may be of sufficient magnitude to cause material failure.

In the preceding practically oriented examples the viscoelastic mechanical properties were slowly varying quantities, and the pressure loading on the boundary and the movement of the eroding boundary were smooth continuous functions. However, the procedure used here would be equally applicable to situations in which these quantities were much less well-behaved functions.

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